Characterizing exact solutions from asymptotic physical concepts

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Abstract

We contribute to the subject of the physical interpretation of exact solutions by characterizing them through a systematic study in terms of unambiguous physical concepts coming from systems in linearized gravity. We use the physical meaning of the leading order behavior of the Weyl spinor components Ψ_0^0, Ψ_1^0 and Ψ_2^0 and of the Maxwell spinor components ϕ_0^0 and ϕ_1^0 and integrate from future null infinity inwards the exact field equations. In this way it is assigned an unambiguous physical meaning to exact solutions and we indicate a method to generalize the procedure to radiating spacetimes.

1 Introduction

The problem of constructing an adequate dynamical description of a system of gravitating compact objects has received much attention recently. Several different approaches can be distinguished to tackle this problem. They range from post-Newtonian schemes, which can handle systems with low relative velocities, to numerical methods that confront the problem of solving the

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full Einstein equations; by evolving some given initial data on a spacelike hypersurface.

There are however other approaches that intend to use mainly an evolution in terms of characteristic surfaces. In the past we have suggested a set of sections at future null infinity($\mathcal{I}^+$), the so called "nice sections"[1], that define asymptotically, in the interior of the spacetime, a set of appropriate characteristic surfaces for this task. Very recently we have proved that this sections exist globally on $\mathcal{I}^+$, and that they have the expected correct behavior[2][3].

In all these approaches it is important to know how to characterize the physical system one has in mind in terms of the physical fields. For example, there is growing interest in the study of collisions of compact objects and the generation of gravitational radiation in these processes. If one tries to describe its evolution in terms of characteristic data, one is faced with the problem of ascribing to the asymptotic physical fields the information of the physical system in the interior of the spacetime.

In this article we would like to contribute on the understanding of our approach by concentrating on the problem of the physical meaning of the asymptotic fields at future null infinity. The point is that if one has a preferred set of sections at $\mathcal{I}^+$, or equivalently a preferred set of characteristic hypersurfaces, one should also know how to read or introduce initial data with the desired physical meaning. We know how to ascribe unambiguous physical meaning to the fields in the context of linearized gravity; however one encounters some difficulties when one wants, for example, to give an unambiguous physical meaning to total angular momentum in a spacetime admitting gravitational radiation[4][5].

One possibility for the study of a system with compact objects is to try to build a framework that uses a finite set of degrees of freedom; which are determined by the finite parameters determining the compact objects. In this approach one would need to know how to read this structure from the physical fields, in particular, the asymptotic geometric fields at future null infinity. More concretely, if the system consists of two compact objects; which are characterized by their respective mass, momentum, intrinsic angular momentum and their relative position; one would need to relate this structure, of the interior of the spacetime, with the structure of the asymptotic fields. The present work is a contribution on this relation.

In some realistic astrophysical systems there is an initial era in which the gravitational radiation is very small; as for example is the case of a coalescent binary system. When the compact objects are very far apart and with small relative velocities, one would like to resort to properties of each compact object. In this regime, the quantities describing these properties, should

have a clear counterpart in the corresponding description of the system in linearized gravity. And it should be noted that the concepts constructed in linearized gravity have a natural extension to non-radiating spacetimes; where the supertranslation gauge freedom can be fixed. The first task is then to understand the relation between the structure of the sources and that of the asymptotic fields when there is no gravitational radiation.

Consequently, in this paper we start to study the issue by restricting ourselves to the case of stationary spacetimes. We construct a relation between the structure of the asymptotic fields, for systems with compact objects, with the structure of the internal fields coming from Einstein equations. More concretely, we integrate "inwards", from $\mathcal{I}^+$, the exact field equations from appropriate asymptotic data. These asymptotic data are suggested by the analysis of the corresponding system in linearized gravity, where one has the unambiguous physical interpretation provided by the background metric.

We make extensive use of the null tetrad formalism[7], following the notation defined by Geroch, Held, and Penrose in [8] (which we refer to as GHP). The explicit form of the spin coefficients, Einstein equations, and the Bianchi identities can be found in [8] and [9]. In the next section we define the null tetrads upon which our calculations are based. Two definitions of tetrads and associated coordinate systems are given for asymptotically flat spacetimes.

In section 3 we analyze the properties of asymptotic fields in linearized gravity. In subsections 3.1 and 3.2 we study the asymptotic structure of the Weyl tensor, and the Maxwell fields respectively.

Motivated by the results of 3.1 we give, in section 4, the definitions of *monopolar spacetimes* and *monopolar spacetimes with angular momentum*. We find all the exact solutions satisfying these definitions.

In section 5 we generalize the results of the previous section to the case of Einstein-Maxwell solutions, and we give the definitions of *monopolar spacetimes with charge* and *monopolar spacetimes with angular momentum and charge* respectively. We also find the exact solutions satisfying these last definitions.

2 Standard tetrads and rotating tetrads of future asymptotically flat space-times

2.1 Standard Tetrad

Our studies are all concerned with asymptotically flat spacetimes at future null infinity. The first task is to define the appropriate null tetrads in terms of

which we express Einstein equations in the GHP formalism. It is convenient to introduce the notion of *standard* and *rotating null tetrads* which are defined in terms of the asymptotic structure of the spacetime.

In Appendix B, we present the basic equations associated to a general null tetrad adapted to an arbitrary null congruence, where it is shown how a local coordinate system can be constructed based on an appropriate hypersurface Σ. In this section we make the analogous construction taking null infinity as our preferred hypersurface. We assume null infinity ($\mathcal{I}^+$) to have the topology of $S^2 \times \mathbb{R}$[9]. Therefore, we can make use of a coordinate system $(\tilde{u}, \tilde{x}^2, \tilde{x}^3)$ on $\mathcal{I}^+$ (where $\tilde{u}$ corresponds to a coordinate along the generators of $\mathcal{I}^+$ and $\tilde{x}^2$, $\tilde{x}^3$ coordinatize the sphere S^2). In this way the equation $\tilde{u} = \mathtt{constant}$ determines a family of smooth sections on $\mathcal{I}^+$ which we shall denote by S_u. Conversely given any smooth family of sections on $\mathcal{I}^+$, labeled by $\tilde{u}$, one can construct the coordinate system $(\tilde{u}, \tilde{x}^2, \tilde{x}^3)$ at $\mathcal{I}^+$, such that the coordinates $(\tilde{x}^2, \tilde{x}^3)$ are constant along the generators of $\mathcal{I}^+$.

Let S_u be a given smooth family of sections of $\mathcal{I}^+$. One can extend the coordinate system $(\tilde{u}, \tilde{x}_2, \tilde{x}_3)$ into the interior of the spacetime by considering the null geodesics in the interior reaching each S_u orthogonally. For each $\widetilde{u}$, this family of null geodesics generates a null surface N_u; we define the function u in the interior of the spacetime such that it is constant on each N_u, and $u = \tilde{u}$ at $\mathcal{I}^+$. Similarly, we define the coordinates (x^2, x^3) in a neighborhood of $\mathcal{I}^+$, such that they are constant along the generators of N_u, and $x^2 = \tilde{x}^2$ and $x^3 = \tilde{x}^3$ at $\mathcal{I}^+$. The fourth coordinate r is taken as an affine parameter along the generators of N_u. In this manner we obtain a suitable coordinate system (u, r, x^2, x^3) in a neighborhood of $\mathcal{I}^+$. By construction this null congruence is surface-forming, i.e., the null vector field $\ell_a = (du)_a$ is tangent to the null geodesics for which r is an affine parameter, i.e., $\ell^a = \left(\frac{\partial}{\partial r}\right)^a$. Latin indices will be understood as abstract indices unless otherwise stated.

Now we construct the null tetrad $(\ell_a, n_a, m_a, \bar{m}_a)$ following the path described in appendix B.

In order to complete the definition of the null tetrad, that we shall call *standard*, additional conditions must be required. First we impose that the vector m^a (and its complex conjugate) satisfies

$$m\,(u) = m\,(r) = 0. \tag{1}$$

This is always possible by means of null rotations of type I and II, see appendix A.

Then we require the coordinate system $(\tilde{u}, \tilde{x}^2, \tilde{x}^3)$ at infinity to be Bondi-like; i.e., such that the restriction of the conformal metric tensor to $\mathcal{I}^+$ is

given by

$$\tilde{g}^{+} = -\frac{4d\zeta d\bar{\zeta}}{\left(1+\zeta\bar{\zeta}\right)^{2}}, \tag{2}$$

where we are using the stereographic coordinates $\zeta = \frac{1}{2}\left(\tilde{x}^2 + i\tilde{x}^3\right)$ on the sphere.

Under a tetrad rotation of the form $m^a \to e^{i\theta} m^a$ the spin coefficient ϵ transforms in the following way:

$$\epsilon \to \epsilon + \frac{1}{2} i\ell\left(\theta\right). \tag{3}$$

The spin coefficient ϵ is pure imaginary, since ℓ^a is geodesic[1]; therefore, it is possible to choose θ such that ϵ vanishes. The second condition is then $\epsilon = 0$. There is still a freedom in the choice of θ, namely, we can add any arbitrary function θ_0 independent of r. This freedom represents a rotation at infinity which allows us to take

$$m^a = \frac{\sqrt{2}P_0}{r}\left(\frac{\partial}{\partial\zeta}\right)^a + O\left(\frac{1}{r^2}\right), \tag{4}$$

where $P_0 = \frac{1}{2}\left(1+\zeta\bar{\zeta}\right)$.

Finally, the affine parameter r is undetermined by an additive function $r_0\left(u, x^2, x^3\right)$; which we fix by the requirement

$$\rho = -\frac{1}{r} + O\left(\frac{1}{r^3}\right); \tag{5}$$

which is equivalent to think of r as the luminosity distance. This choice of r not only simplifies ρ, but also gives it physical relevance.

The leading terms X_0^2 and X_0^3 (which are independent of r), of the asymptotic expansion of the components of the null vector n^a (see (45)), vanish because of the definition of the asymptotic coordinates and the fact that the vector field n^a is tangent to $\mathcal{I}^+$. It can be seen that X_0^0 has the unit value in this case; due to the fact that $n^a \ell_a = 1$ and $\ell = du$[9].

In appendix B we present the equations imposed by the torsion-free and metric conditions on the connection for a general tetrad associated to a null congruence. The corresponding relations for a *standard tetrad* are easily obtained by setting $\omega = 0$, $\xi^0 = 0$, $\rho = \bar{\rho}$, $\kappa = 0$, and $\epsilon = 0$ in those equations.

[1] The real part of ϵ is proportional to $n^a \ell^b \nabla_b l_a$.(see definition of the spin coefficients in [8])

2.2 Rotating Tetrad

Although it is always possible to construct a *standard tetrad* in a neighborhood of null infinity for any asymptotically flat spacetime, in certain cases it will be more convenient to work with null tetrads adapted to a twisting congruence of null geodesics. In this section we define the notion of *rotating tetrad*, as a particularization of the general tetrad given in appendix B.

The fundamental condition imposed on the *rotating tetrad* is the vanishing of the leading asymptotic term of one of the Weyl tensor components, namely

$$\lim_{r\to\infty} r^4\psi_1 \equiv \psi_1^0 = 0, \tag{6}$$

where $\psi_1 = C_{abcd}\ell^a n^b \ell^c m^d$. This can be achieved by noting that at the section S_u one can construct a congruence of null geodesics such that $\psi_1 = 0$ on S_u by appropriately choosing the corresponding twist of the congruence at $\mathcal{I}^+$. The resulting congruence fails to be surface forming ($\rho - \bar{\rho} \neq 0$). Additional conditions must be imposed in order to completely determine the tetrad and it associated coordinate system.

As in the case of the *standard tetrad* we take Bondi-coordinates $(\tilde{u}, \tilde{x}_2, \tilde{x}_3)$ at $\mathcal{I}^+$. The 'origin' of the coordinate r (affine parameter along the twisting geodesics) it is chosen in such a way that

$$\rho + \bar{\rho} = -\frac{2}{r} + O\left(\frac{1}{r^3}\right). \tag{7}$$

This condition is analogous to the one imposed to the *standard tetrad* in eq. (5).

It is again possible to choose the vector m^a satisfying $m\,(r) = 0$ (see appendix A). In this case the tetrad can also be chosen such that $\epsilon = 0$, in an analogous way as in the previous case for the *standard tetrad*. The remaining freedom encoded in θ_0, in the transformation $m^a \to e^{i\theta} m^a$, is fixed by requiring

$$\xi_0^2 = \frac{\sqrt{2}P_0}{r} + O\left(\frac{1}{r^2}\right), \tag{8}$$

$$\xi_0^3 = -i\frac{\sqrt{2}P_0}{r} + O\left(\frac{1}{r^2}\right); \tag{9}$$

which is equivalent to condition (4). From the definition of the asymptotic coordinate system, one can show that the leading terms in the asymptotic expansion of the components of n^a take the values

$$X_0^0 = 1, \quad X_0^2 = 0, \quad X_0^3 = 0, \tag{10}$$

for any asymptotically flat spacetime. The torsion-free and metric conditions for the *rotating tetrad* can be obtained, from the general equations appearing in the appendix B by setting $\omega = 0$, $\kappa = 0$, and $\epsilon = 0$. Finally let us point out that the notion of *rotating tetrad* exists for any asymptotically flat spacetime, and coincides with that of the *standard tetrad* when ψ_1^0 vanishes.

3 Linearized gravity

3.1 The Weyl Tensor in Stationary Asymptotically Flat Space-times (Vacuum case).

In this section we analyze the asymptotic structure of the Weyl tensor in linearized gravity for isolated systems. The results obtained here will provide physical motivation for the asymptotic conditions required by the definitions that will be made in sections 4 and 5. The study is based on the restrictions imposed on the curvature by the Bianchi identities. These restrictions define the multipolar structure of the geometry of an isolated system in linearized gravity.

We use a null coordinate system, and the corresponding null tetrad ($\ell^a, n^a, m^a, \bar{m}^a$) adapted to a congruence of hypersurface-orthogonal, shear free, null geodesics of Minkowski spacetime. These conditions imply that this null coordinate system is the one defined by the null cones emanating from an arbitrary world line $\gamma(u)$ given by the equation $x^\mu = z^\mu(u)$; where $\mu = 0, 1, 2, 3$. In our case we take $z^\mu(u) = u\delta_0^\mu$. The usual Cartesian coordinates $(x^\mu) = (t, x, y, z)$ are related to the null polar coordinates $\left(u, r, \zeta, \bar{\zeta}\right)$ by $x^\mu = z^\mu(u) + r\, l^\mu\left(\zeta, \bar{\zeta}\right)$, where $l^\mu\left(\zeta, \bar{\zeta}\right)$ is the null vector in Minkowski spacetime pointing in the direction given by ζ and $\bar{\zeta}$, where $(l^\mu) = \frac{1}{2P_0}(1 + \bar{\zeta}\zeta, \zeta + \bar{\zeta}, i(\bar{\zeta} - \zeta), -1 + \zeta\bar{\zeta})$. In terms of the above coordinate system the line element is $ds^2 = du^2 + 2dudr - r^2\frac{d\zeta d\bar{\zeta}}{P_0^2}$, and the null tetrad becomes:

$$\ell^a = \left(\frac{\partial}{\partial r}\right)^a, \tag{11}$$

$$n^a = \left(\frac{\partial}{\partial u}\right)^a - \frac{1}{2}\left(\frac{\partial}{\partial r}\right)^a, \tag{12}$$

$$m^a = \frac{\sqrt{2}P_0}{r}\left(\frac{\partial}{\partial \zeta}\right)^a. \tag{13}$$

The non-vanishing spin coefficients corresponding to our null tetrad are: $\rho_M = -\frac{1}{r}$, $\rho'_M = \frac{1}{2r}$, and $\beta_M = \bar{\beta}'_M = -\frac{1}{r\sqrt{2}}\frac{\partial P}{\partial \zeta}$.

We assume that the energy-momentum tensor of linearized gravity is stationary and has compact support; therefore, the field equations have no sources in a neighborhood of $\mathcal{I}^+$.

The linearized Bianchi identities depend only on the spin coefficients of Minkowski spacetime. In a regular asymptotically flat spacetime[9] the Weyl components ψ_j with $(j = 0, 1, 2, 3, 4)$ are given by the following asymptotic series in negative powers of r

$$\psi_j = \sum_{n=0} \frac{\psi_j^n}{r^{(n+5-j)}}, \tag{14}$$

where the ψ_j^n are independent of r.

The radial Bianchi identities, (see equations (3.88-3.91) in [9]), restrict the asymptotic series in r for the components of the Weyl tensor to:

$$\psi_1 = \frac{\psi_1^0}{r^4} - \frac{\overline{\eth}_0 \psi_0^0}{r^5} - \frac{1}{2}\frac{\overline{\eth}_0 \psi_0^1}{r^6} - \frac{1}{3}\frac{\overline{\eth}_0 \psi_0^2}{r^7} - \frac{1}{4}\frac{\overline{\eth}_0 \psi_0^3}{r^8} + \ldots, \tag{15}$$

$$\psi_2 = \frac{\psi_2^0}{r^3} - \frac{\overline{\eth}_0 \psi_1^0}{r^4} + \frac{1}{2}\frac{\overline{\eth}_0^2 \psi_0^0}{r^5} + \frac{1}{2\times 3}\frac{\overline{\eth}_0^2 \psi_0^1}{r^6} + \frac{1}{3\times 4}\frac{\overline{\eth}_0^2 \psi_0^2}{r^7} + \ldots, \tag{16}$$

$$\psi_3 = \frac{\psi_3^0}{r^2} - \frac{\overline{\eth}_0 \psi_2^0}{r^3} + \frac{1}{2}\frac{\overline{\eth}_0^2 \psi_1^0}{r^4} - \frac{1}{2\times 3}\frac{\overline{\eth}_0^3 \psi_0^0}{r^5} + \frac{1}{2\times 3\times 4}\frac{\overline{\eth}_0^3 \psi_0^1}{r^6} + \ldots, \tag{17}$$

$$\psi_4 = \frac{\psi_4^0}{r} - \frac{\overline{\eth}_0 \psi_3^0}{r^2} + \frac{1}{2}\frac{\overline{\eth}_0^2 \psi_2^0}{r^3} - \frac{1}{2\times 3}\frac{\overline{\eth}_0^3 \psi_1^0}{r^4} + \frac{1}{2\times 3\times 4}\frac{\overline{\eth}_0^4 \psi_0^0}{r^5} + \ldots, \tag{18}$$

where the operators $\eth_0$, and $\overline{\eth}_0$ represent the edth operators on the unit sphere[13][10] in the GHP notation[8].

New restrictions on the Weyl components come from the non-radial Bianchi identities, corresponding to the primed eqs.(3.88-3.93) in [9]. These restrictions fix the angular dependence of the Weyl components in the following way:

$$\psi_0 = \sum_{\ell=2} \sum_{m=-\ell}^{\ell} \frac{a^{\ell m}}{r^{\ell+3}} \, {}_2Y_{lm}, \tag{19}$$

$$\psi_1^0 = \sum_{m=-1}^{1} b^m \, {}_1Y_{1m}, \tag{20}$$

$$\psi_2^0 = -M, \tag{21}$$

where the ${}_sY_{\ell m}$ are generalized spherical harmonics of spin weight s[13], and $a^{\ell m}$, b^m together with M are constants. From the stationarity requirement and the Bianchi identities one can see that the remaining leading order terms ψ_3^0 and ψ_4^0 vanish[9].

This constitutes the multipolar structure of stationary isolated systems in linearized gravity.

3.2 The Electromagnetic Tensor

In linearized Einstein-Maxwell theory Maxwell equations are directly solved on the flat Minkowski background. In this section we recall the well known result of the multipolar decomposition of the electromagnetic field in Minkowski spacetime in terms of the null tetrad formalism. The results will be used in section 5 to define the asymptotic data of the Maxwell fields for the exact Einstein-Maxwell equations.

The components of the electromagnetic tensor in the null tetrad are $\phi_0 \equiv F_{ab}\ell^a m^b$, $\phi_1 \equiv \frac{1}{2}F_{ab}\left(\ell^a n^b + \bar{m}^a m^b\right)$ and $\phi_2 \equiv F_{ab}\bar{m}^a n^b$. The radial dependence of the electromagnetic field of compact sources is determined by the un-primed Maxwell equations in the Newman-Penrose formalism[8][15]. If we denote the asymptotic series of ϕ_0 by

$$\phi_0 = \sum \frac{\phi_0^i}{r^{3+i}}, \tag{22}$$

then the other components are given by:

$$\phi_1 = \frac{q}{r^2} - \frac{\bar{\eth}_0\phi_0^0}{r^3} - \frac{1}{2}\frac{\bar{\eth}_0\phi_0^1}{r^4} - \frac{1}{3}\frac{\bar{\eth}_0\phi_0^2}{r^5}\cdots, \tag{23}$$

$$\phi_2 = \frac{\phi_2^0}{r} - \frac{\bar{\eth}_0\phi_1^0}{r^2} + \frac{1}{2}\frac{\bar{\eth}_0^2\phi_0^1}{r^3} + \frac{1}{6}\frac{\bar{\eth}_0^2\phi_0^2}{r^4}\cdots \tag{24}$$

Finally, the solution of all the Maxwell equations for the case of stationary compact sources has the following structure:

$$\phi_0 = \frac{\sum \mu^m \, {}_1Y_{1m}}{r^3} + \frac{\sum Q^m \, {}_1Y_{2m}}{r^3} + \ldots, \tag{25}$$

$$\phi_1 = \frac{q}{r^2} + \dots , \tag{26}$$

$$\phi_2 = O\left(\frac{1}{r^3}\right), \tag{27}$$

where μ^m, Q^m, and q are constants, corresponding to the dipolar moment, quadrupolar moment, and charge of the Maxwell field respectively. The component ϕ_2^0 corresponds to electromagnetic radiation. In the stationary case one has $\phi_2^0 = 0$.

4 Vacuum exact solutions

In section 3.1 we studied the asymptotic structure of the Weyl tensor required by the Bianchi identities in linearized gravity. The structure of a compact object in linearized gravity appears in the asymptotic expansion of the Weyl tensor as a multipolar series in inverse powers of the radial coordinate. Now we look for exact solutions of Einstein equations that have the analogous asymptotic structure. The vacuum exact solutions that are characterized by the restrictions imposed by the asymptotic structure of compact sources in linearized gravity are given by the following definitions.

4.1 Monopolar spacetime

Definition 1 *We call a spacetime* monopolar *if it is a stationary, vacuum asymptotically flat, solution of Einstein equations and, in addition, the following conditions hold in the* standard tetrad *defined in section (2.1), namely:* $\psi_0 = \psi_1^0 = 0$, *and* $\psi_2^0 \neq 0$.

The conditions $\psi_0 = 0$ and $\psi_1^0 = 0$ correspond to the requirement that the spacetime does not have further structure than mass (recall that according to our results of section (3.1) in linearized gravity ψ_0 is where quadrupolar and higher momenta are encoded, while ψ_1^0 corresponds to the angular momentum data).

It turns out that the definition of monopolar spacetimes is strong enough to single out a family of solutions of the Einstein equations. The result is expressed in the following theorem.

Theorem 1 The solutions to the vacuum Einstein equations which are stationary, asymptotically flat, and in the *standard tetrad* $\psi_0 = \psi_1^0 = 0$, and $\psi_2^0 \neq 0$, are given by the one-parameter family of Schwarzschild spacetimes. Alternatively, a

spacetime is *monopolar*, in the sense given by the definition above, if and only if it belongs to the one-parameter family of Schwarzschild geometries.

Proof: We start with Sachs equations, the optical scalars equations in the GHP formalism, that in this case reduce to:

$$\frac{\partial\rho}{\partial r} = \rho^2 + \sigma\bar{\sigma} \quad , \quad \frac{\partial\sigma}{\partial r} = 2\rho\sigma. \tag{28}$$

This set of equations are solved by

$$\sigma = \frac{\sigma_0}{r^2 - \sigma_0\bar{\sigma}_0} \quad , \quad \rho = \frac{-r}{r^2 - \sigma_0\bar{\sigma}_0}; \tag{29}$$

where σ_0 does not depend on r.

The Bianchi identity involving the radial derivative of ψ_1[9] reduces to

$$\frac{\partial\psi_1}{\partial r} = 4\rho\psi_1; \tag{30}$$

which implies that $\psi_1 = \frac{\psi_1^0}{(r^2-\sigma_0\bar{\sigma}_0)^2}$. The definition of monopolar spacetime requires that $\psi_1^0 = 0$; therefore, ψ_1 must vanish. At this point, the hypothesis of the Goldberg-Sachs theorem are fulfilled, and consequently the shear σ must vanish. This implies that the spacetime is shear free algebraically special of type II. The only vacuum shear free, algebraically special solution of type II are the Robinson-Trautman spacetimes[14]; and among the Robinson-Trautman solutions the only stationary ones are the Schwarzschild geometries.$\Box$

4.2 Monopolar spacetime with angular momentum

The next step is to introduce angular momentum in our definitions. The existence of angular momentum is given, in the *standard tetrad*, by a non-vanishing ψ_1^0. As we pointed out in our definitions of null tetrads in this case we have the possibility of defining a spacetime with angular momentum in either the standard tetrad or the rotating tetrad, since they differ when $\psi_1^0 \neq 0$. In this section we study the implications of both alternatives in the following definitions of monopolar spacetimes with angular momentum.

Definition 2 *A spacetime is called* monopolar with angular momentum (type a) *when it is a stationary, vacuum, locally asymptotically flat*[2] *solution of Einstein equations, and in addition, its Weyl tensor in the* standard tetrad *satisfies that* $\psi_1^0 \neq 0,\ \psi_2^0 \neq 0,\ \psi_0 = 0.$

[2]By locally asymptotically flat it is meant that the conditions for asymptotic flatness of [9] are satisfied locally at $\mathcal{I}^+$. In particular it might be that future null infinity is not complete.

Starting with the asymptotic conditions imposed by the previous definition one proceeds to integrate inwards the Einstein equations from future null infinity. The result of these calculations are presented in the following theorem.

Theorem 2 The spacetimes which are *monopolar with angular momentum (type a)* are the stationary Newman-Tamburino solutions.

Proof: One can easily see that the definition 2a implies

$$\psi_0 = \rho - \bar{\rho} = 0, \tag{31}$$

$$\rho^2 \neq \sigma\bar{\sigma} \quad , \quad \rho \neq 0, \tag{32}$$

$$\nabla_{[a}\ell_{b]} = 0, \tag{33}$$

$$\psi_1^0 \neq 0. \tag{34}$$

These equations constitute the complete characterization of the Newman-Tamburino solutions[14]. Since the definition requires the spacetime to be stationary, one concludes that the spacetimes which are *monopolar with angular momentum of type a* are determined by the stationary Newman-Tamburino solutions.

These solutions are not asymptotically flat in all null future directions and therefore do not describe the type of physical systems in which we are interested. This result suggests that the twist-free *standard tetrad* might not be well suited to the study of spacetime geometries with angular momentum. In the following definition we apply our strategy making use of the *rotating tetrad*.

Definition 3 *A spacetime is called* monopolar with angular momentum (type b) *when it is a stationary, vacuum, locally asymptotically flat solution of Einstein equations, and, in addition, its Weyl tensor in the* rotating tetrad *satisfies that* $\psi_2^0 \neq 0$ *and* $\psi_0 = 0$[3].

These spacetimes are characterized by:

[3]In both definitions we take $\psi_0 = 0$ as we did in Definition 1 in order to introduce no further multipole moments.

Theorem 3 The spacetimes which are *monopolar with angular momentum (type b)* correspond to the spherical Kerr family. The following are the expressions for the components of the null tetrad.

$$\xi^i = \frac{\xi_0^i}{(r - ia\cos(\theta))}, \tag{35}$$

$$X^i = \delta_0^i - \frac{\left(\tau_0 \bar{\xi}_0^i + \bar{\tau}_0 \xi_0^i\right)}{(r^2 + a^2\cos^2(\theta))}, \tag{36}$$

$$U = -\frac{1}{2} - \frac{\tau_0 \bar{\tau}_0}{(r^2 + a^2\cos^2(\theta))} + \frac{Mr}{(r^2 + a^2\cos^2(\theta))}, \tag{37}$$

with $\xi_0^0 = -\tau_0 = i\eth_0\left(a\cos(\theta)\right)$. The parameters M and a are the mass and angular momentum parameter respectively.

We refer the reader to the Appendix C where this theorem is proved.

5 Einstein-Maxwell exact solutions

Our construction for vacuum spacetimes can be generalized to the electrovac case using the same idea of giving asymptotic data defined by the study of fields in the linear theory where their physical interpretation is available. In section 3.2 we have described the structure of the asymptotic electromagnetic field in terms of a null tetrad. Previously we have showed, in the vacuum case, that imposing analogous asymptotic conditions we can single out certain families of exact solutions. In this section we continue with the program, and combining the linear analysis of Weyl and electromagnetic tensors we give new definitions of Einstein-Maxwell space-times which generalizes the previous results.

In the following definition we start by adding charge to the Monopolar spacetimes.

Definition 4 *We define a spacetime to be* monopolar with charge *when it is a stationary asymptotically flat solution of Einstein-Maxwell equations such that in the* standard tetrad *its Weyl and electromagnetic tensor satisfy the following asymptotic conditions:* $\psi_0 = \psi_1^0 = 0$, $\psi_2^0 \neq 0$ *and* $\phi_0 = 0$, $\phi_1^0 \neq 0$.

The condition $\phi_0 = 0$ rules out higher multipole moments in the electromagnetic field in the spirit of the linearized analysis.

The restrictions imposed by the previous definitions are reflected by the following theorem.

Theorem 4 The stationary, asymptotically flat solutions of the Einstein-Maxwell equations such that in the *standard tetrad* $\psi_0 = \psi_1^0 = 0$, $\psi_2^0 \neq 0$, $\phi_0 = 0$, and $\phi_1^0 \neq 0$ are given by the two-parameter family of Reissner-Nordstrom spacetimes. Alternatively, using the definition given above, a spacetime is *monopolar with charge* if and only if it belongs to the Reissner-Nordstrom family.

We will give a full proof of the following theorem which generalizes the previous one. In the sequel we introduce charge to the Monopolar spacetimes with angular momentum.

Definition 5 *We define a spacetime to be monopolar with charge and angular momentum when it is a stationary asymptotically flat solution of Einstein-Maxwell equations such that in the* rotating tetrad *its Weyl and electromagnetic tensor satisfy the following asymptotic conditions:* $\psi_0 = 0$, $\phi_0 = 0$, *and* $\phi_1 \neq 0$.

These spacetimes are characterized by the following theorem.

Theorem 5 The stationary, asymptotically flat solutions of the Einstein-Maxwell equations such that in the *rotating tetrad* $\psi_0 = 0$, and $\phi_0 = 0$ are given by the three-parameter family of Kerr-Newman spacetimes. Alternatively, a spacetime is *monopolar with charge and angular momentum* if and only if it belongs to the Kerr-Newman family. The following components of the null tetrad are given by

$$\xi^i = \frac{\xi_0^i}{(r - ia\cos(\theta))}, \tag{38}$$

$$X^i = \delta_0^i - \frac{\left(\tau_0\bar{\xi}_0^i + \bar{\tau}_0\xi_0^i\right)}{(r^2 + a^2\cos^2(\theta))}, \tag{39}$$

$$U = -\frac{1}{2} - \frac{(\tau_0\bar{\tau}_0 + q\bar{q})}{(r^2 + a^2\cos^2(\theta))} + \frac{Mr}{(r^2 + a^2\cos^2(\theta))}, \tag{40}$$

with $\xi_0^0 = -\tau_0 = i\eth_0\left(a\cos(\theta)\right)$. The parameters M, a, and q are the mass, angular momentum parameter, and charge respectively.

We refer the reader to the second part of the Appendix C where this theorem is proved.

Higher Momenta: In all the previous definitions we have required the vanishing of the Weyl component ψ_0. The motivation for these requirement comes from the study in linearized gravity of the multipolar structure of compact sources. In section 3 we showed that in the linear vacuum theory ψ_0 contains information concerning quadrupolar and higher momenta of the gravitational field. In the spirit of this work, the condition $\psi_0 = 0$ was imposed in order to exclude the possibility of having higher momenta, and in this way, to give a model of particle in general relativity. One can show that in the linearized Einstein-Maxwell problem it is possible to include charge without breaking this structure (as we did in the previous definitions). However, if one tries to introduce higher multipolar terms in the Maxwell field one has to necessarily abandon the condition $\psi_0 = 0$. For example, in linearized gravity, if one includes a dipolar momentum μ in the Maxwell field (i.e., if one takes $\phi_1 = \frac{\mu}{r^3}$) then the Bianchi identities require for ψ_0 that $3\psi_0^1 + \bar{\eth}_0 \eth_0 \psi_0^1 = -2\eth_0 \mu \eth_0 \bar{\mu}$. In order to include a dipolar Maxwell field one has to necessarily include higher momenta in the gravitational field.

6 Final comments

We have succeeded in obtaining exact solutions of Einstein equations starting with asymptotic data with a natural physical interpretation in linearized gravity. These data were designed to mimic the structure of well understood solutions of linearized gravity representing compact objects, where the presence of the Minkowski background metric provides means of unambiguous physical interpretation.

In the description of systems with angular momentum the results of section 2.2 show the convenience of the notion of the *rotating tetrad* (a twisting null tetrad chosen to annihilate the leading asymptotic component of ψ_1).

It is striking that the physically most relevant stationary Einstein-Maxwell solutions representing isolated systems are recovered by means of our method.

Finally, this work suggests the possibility of generalizing the method to non-stationary asymptotically flat spacetimes. As it is shown in reference [2] there is a preferred family of sections of $\mathcal{I}^+$, the *nice sections*, that represent the notion of instantaneous rest frame for asymptotically flat spacetimes. We would like to extend our construction to non-stationary cases by basing the null tetrads on these family of sections. In this way we expect to be able to contribute to the construction of an approximation scheme, suitable for the description of the dynamics of compact objects in general relativity, by means of a model with a finite number of degrees of freedom (mass, angular momentum, etc.).

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A Tetrad transformations

The freedom in choosing a null tetrad satisfying the normalization conditions $\ell^a n_a = -m^a \overline{m}_a = 1$, where all other contractions vanish, is given by the following transformations[16]:

Transformation of type I

$$\ell^a \to \ell^a,$$

$$m^a \to m^a + \Gamma \ell^a,$$

$$n^a \to n^a + \Gamma \overline{m}^a + \overline{\Gamma} m^a + \Gamma \overline{\Gamma} \ell^a.$$

Transformation of type II

$$n^a \to n^a,$$

$$m^a \to m^a + \Lambda n^a,$$

$$\ell^a \to \ell^a + \Lambda \overline{m}^a + \overline{\Lambda} m^a + \Lambda \overline{\Lambda} n^a.$$

Transformation of type III

$$\ell^a \to Z \ell^a,$$

$$n^a \to Z^{-1} n^a,$$

$$m^a \to e^{i\theta} m^a,$$

where Γ and Λ are complex scalars, whereas Z and θ are real. Therefore, we have six real scalar representing the degrees of freedom of the local Lorentz rotations.

B General tetrad associated to a null congruence

This appendix contains the equations relating the spin coefficients, in the GHP formalism, with the null tetrad components associated with the coordinate system adapted to a real null vector field ℓ^a, in the sense explained below. The equations corresponding to the null tetrads used throughout this paper can be simply obtained by setting certain coefficients to zero according to the following rules. In order to obtain the equations in the *standard tetrad* of section 2.1 set $\omega = \xi^0 = \rho - \bar{\rho} = \kappa = \epsilon = 0$ in the equations for the general tetrad presented in this appendix. Set $\omega = \kappa = \epsilon = 0$ to obtain the relations among spin coefficients in the *rotating tetrad*.

The coordinate r is chosen such that the vector field ℓ^a is given by:

$$\ell^a = \left(\frac{\partial}{\partial r}\right)^a . \tag{41}$$

Let Σ be a hypersurface such that the integral lines of the vector field ℓ^a intercept Σ only once. Let $\left\{\hat{t}, \hat{x}^2, \hat{x}^3\right\}$ be a coordinate system on Σ; then, in a neighborhood of Σ one can define coordinates $\{\, t, r, x^2, x^3\}$ such that on Σ

$$t = \hat{t} \quad , \quad x^2 = \hat{x}^2 \quad , \quad x^3 = \hat{x}^3, \tag{42}$$

and $\{\, t, x^2, x^3\}$ are constant along the integral lines of ℓ^a.

In this coordinate system the remaining three null vectors are given by:

$$m^a = \omega \left(\frac{\partial}{\partial r}\right)^a + \xi^i \left(\frac{\partial}{\partial x^i}\right)^a , \tag{43}$$

$$\bar{m}^a = \bar{\omega} \left(\frac{\partial}{\partial r}\right)^a + \bar{\xi}^i \left(\frac{\partial}{\partial x^i}\right)^a , \tag{44}$$

$$n^a = U \left(\frac{\partial}{\partial r}\right)^a + X^i \left(\frac{\partial}{\partial x^i}\right)^a , \tag{45}$$

where the index i takes the values $i = 0, 2, 3$; the vector n^a is real and $\bar{m}^a$ represents the complex conjugate of m^a.

The corresponding representation of the dual null tetrad is given by:

$$\ell_a = \frac{1}{d} \epsilon_{ijk} \xi^j \bar{\xi}^k \left(dx^i\right)_a , \tag{46}$$

$$m_a = \frac{1}{d}\epsilon_{ijk}\xi^j X^k \left(dx^i\right)_a, \tag{47}$$

$$n_a = \frac{1}{d}\epsilon_{ijk}\left[U\bar{\xi}^j\xi^k + \omega X^j\bar{\xi}^k + \bar{\omega}\xi^j X^k\right]\left(dx^i\right)_a + (dr)_a, \tag{48}$$

where $d \equiv \epsilon_{ijk}X^i\xi^j\bar{\xi}^k$, $\epsilon_{ijk} = \epsilon_{[ijk]}$ with $i, j, k = 0, 2, 3$, and $\epsilon_{023} = 1$.

The torsion free condition on the covariant derivative relates the commutator of the tetrad vectors and the spin coefficients[4], which are:

$$[\ell, m]^a = (\bar{\rho} + \epsilon - \bar{\epsilon})\, m^a + \sigma\bar{m}^a + \left(\bar{\beta}' - \beta - \bar{\tau}'\right)\ell^a - \kappa n^a, \tag{49}$$

$$[\ell, n]^a = (\epsilon' + \bar{\epsilon}')\,\ell^a + (\tau - \bar{\tau}')\,\bar{m}^a + (\bar{\tau} - \tau')\, m^a - (\epsilon + \bar{\epsilon})\, n^a, \tag{50}$$

$$[n, m]^a = (-\epsilon' + \bar{\epsilon}' + \rho')\, m^a + \bar{\sigma}'\bar{m}^a - \bar{\kappa}'\ell^a + \left(\beta - \bar{\beta}' - \tau\right) n^a, \tag{51}$$

$$[m, \bar{m}]^a = (\rho - \bar{\rho})\, n^a + \left(\bar{\rho}^{'} - \rho^{'}\right)\ell^a + \left(\bar{\beta} + \beta'\right) m^a - \left(\beta + \bar{\beta}'\right)\bar{m}^a. \tag{52}$$

Writing these expressions in terms of the coordinate basis, one obtains the following equations:

$$U_r = \left(\epsilon^{'} + \bar{\epsilon}^{'}\right) + \left[\left(\tau - \bar{\tau}^{'}\right)\bar{\omega} + \left(\bar{\tau} - \tau^{'}\right)\omega\right] - U\,(\epsilon + \bar{\epsilon}), \tag{53}$$

$$X^i_r = \left(\tau - \bar{\tau}^{'}\right)\bar{\xi}^i + \left(\bar{\tau} - \tau^{'}\right)\xi^i - (\epsilon + \bar{\epsilon})\, X^i, \tag{54}$$

$$\omega_r = (\bar{\rho} + \epsilon - \bar{\epsilon})\,\omega + \sigma\bar{\omega} + \left(\bar{\beta}^{'} - \beta - \bar{\tau}^{'}\right) - \kappa U, \tag{55}$$

$$\xi^i_r = (\bar{\rho} + \epsilon - \bar{\epsilon})\,\xi^i + \sigma\bar{\xi}^i - \kappa X^i, \tag{56}$$

[4]The definition of the spin coefficients in terms of the covariant derivative of the elements of the null tetrad can be found in ref. [8].

$$U\omega_r + X^k\omega_k - \omega U_r - \xi^k U_k = \\ \left(\beta - \bar{\beta}' - \tau\right) U - \bar{\kappa}' - \left(\epsilon' - \bar{\epsilon}' - \rho'\right)\omega + \bar{\sigma}'\bar{\omega}, \quad (57)$$

$$U\xi^i_r + X^k\xi^i_k - \omega X^i_r - \xi^k X^i_k = \\ \left(\beta - \bar{\beta}' - \tau\right) X^i - \left(\epsilon' - \bar{\epsilon}' - \rho'\right)\xi^i + \bar{\sigma}'\bar{\xi}^i \quad (58)$$

$$\omega\bar{\omega}_r + \xi^p\bar{\omega}_p - \bar{\omega}\omega_r - \bar{\xi}^p\omega_p = \\ \left(\rho - \bar{\rho}\right) U + \left(\bar{\rho}' - \rho'\right) + \left(\bar{\beta} + \beta'\right)\omega - \left(\beta + \bar{\beta}'\right)\bar{\omega} \quad (59)$$

$$\omega\bar{\xi}^i_r + \xi^p\bar{\xi}^i_p - \bar{\omega}\xi^i_r - \bar{\xi}^p\xi^i_p = \\ \left(\rho - \bar{\rho}\right) X^i + \left(\bar{\beta} + \beta'\right)\xi^i - \left(\beta + \bar{\beta}'\right)\bar{\xi}^i, \quad (60)$$

where the sub-indices represent the corresponding coordinate derivative. The previous equations are the ones that allow us to integrate the components of the null tetrad once we have found the spin coefficients that solve Einstein equations in the GHP formalism.

Putting the spin coefficients in terms of the tetrad components, one obtains:

$$\rho = \frac{1}{2d}\left[d_r + \epsilon_{ijk}\left[\left(\omega\bar{\xi}^i_r + \xi^l\bar{\xi}^i_l - \bar{\omega}\xi^i_r - \bar{\xi}^l\xi^i_l\right) - X^i_r\right]\xi^j\bar{\xi}^k\right], \quad (61)$$

$$\sigma = -\frac{\epsilon_{ijk}}{d}\xi^i_r\,\xi^j\,X^k, \quad (62)$$

$$\tau = \frac{\epsilon_{ijk}}{2d}\left[X^i_r\left(X^j\xi^k + \omega\xi^j\bar{\xi}^k\right) + \\ \xi^i_r X^j\left(\bar{\omega}\xi^k - \omega\bar{\xi}^k\right) + \left(\xi^l X^i_l - X^l\xi^i_l\right)\xi^j\bar{\xi}^k\right] - \frac{\omega_r}{2}, \quad (63)$$

$$\kappa = -\frac{\epsilon_{ijk}}{d}\xi^i_r\xi^j\bar{\xi}^k, \quad (64)$$

$$\rho' = -\frac{\epsilon_{ijk}}{2d}\left\{\left(\omega\bar{\xi}^i_r + \xi^l\bar{\xi}^i_l - \bar{\omega}\,\xi^i_r - \bar{\xi}^l\xi^i_l\right)\times \\ \left(U\bar{\xi}^j\xi^k + \omega X^j\bar{\xi}^k + \bar{\omega}\xi^j X^k\right) + \\ \left[\left(-U\xi^i_r - X^l\xi^i_l + \omega X^i_r + \xi^l X^i_l\right)\bar{\xi}^j X^k + \\ \left(U\bar{\xi}^i_r + X^l\bar{\xi}^i_l - \bar{\omega}X^i_r - \bar{\xi}^l X^i_l\right)\xi^j X^k\right]\right\} \\ - \frac{1}{2}\left(\omega\bar{\omega}_r + \xi^l\bar{\omega}_l - \bar{\omega}\omega_r - \bar{\xi}^l\omega_l\right), \quad (65)$$

$$\sigma' = \frac{1}{d}\epsilon_{ijk}\left(U\bar{\xi}^i_r + X^l\bar{\xi}^i_l - \bar{\omega}X^i_r - \bar{\xi}^l X^i_l\right)\bar{\xi}^j X^k, \tag{66}$$

$$\tau' = \frac{\epsilon_{ijk}}{2d}\left[X^i_r\left(X^j\bar{\xi}^k - \bar{\omega}\bar{\xi}^j\xi^k\right) + \right. \\ \left. \xi^i_r X^j\left(\bar{\omega}\xi^k - \omega\bar{\xi}^k\right) + \left(\bar{\xi}^l X^i_l - X^l\bar{\xi}^i_l\right)\bar{\xi}^j\xi^k\right] - \frac{\bar{\omega}_r}{2}, \tag{67}$$

$$\kappa' = \bar{\omega}U_r + \bar{\xi}^k U_k - \left(U\bar{\omega}_r + X^k\bar{\omega}_k\right) + \\ \frac{\epsilon_{ijk}}{d}\left\{\left[X^i_r\bar{\omega} + X^i_l\bar{\xi}^l - \bar{\xi}^i_r U - \bar{\xi}^i_l X^l\right]\right. \\ \left.\left[X^j\left(\omega\bar{\xi}^k - \bar{\omega}\xi^k\right) + \bar{\xi}^j\xi^k U\right]\right\}, \tag{68}$$

$$\beta = \frac{1}{4d}\epsilon_{ijk}\Big[X^i_r\left(X^j\xi^k - \xi^j\bar{\xi}^k\omega\right) - X^i_l\xi^l\xi^i\bar{\xi}^k \\ + \xi^i_r\left(3X^j\xi^k\bar{\omega} - X^j\bar{\xi}^k\omega + 2\xi^j\bar{\xi}^k U\right) + \xi^i_l\left(2X^j\bar{\xi}^l\xi^k + X^l\xi^j\bar{\xi}^k\right) \\ -2\bar{\xi}^i_r X^j\xi^k\omega - 2\bar{\xi}^i_l\xi^l X^j\xi^k\Big] - \frac{\omega_r}{4}, \tag{69}$$

$$\beta' = \frac{1}{4d}\epsilon_{ijk}\Big[X^i_r\left(X^j\bar{\xi}^k - \bar{\xi}^j\xi^k\bar{\omega}\right) - X^i_l\bar{\xi}^l\bar{\xi}^j\xi^k \\ + 2\xi^i_r X^j\bar{\xi}^k\bar{\omega} + 2\xi^i_l\bar{\xi}^l X^j\bar{\xi}^k \\ +\bar{\xi}^i_r\left(2\bar{\xi}^j\xi^k U - X^j\xi^k\bar{\omega} - X^j\bar{\xi}^k\omega\right) + \bar{\xi}^i_l\left(X^l\bar{\xi}^j\xi^k - 2\xi^l X^j\bar{\xi}^k\right)\Big] + \frac{\bar{\omega}_r}{4}, \tag{70}$$

$$\epsilon = \frac{\epsilon_{ijk}}{4d}\Big[-2X^i_r\xi^j\bar{\xi}^k - \xi^i_r\left(X^j\bar{\xi}^k + \xi^j\bar{\xi}^k\bar{\omega}\right) - \xi^i_l\bar{\xi}^l\xi^j\bar{\xi}^k \\ -\bar{\xi}^i_r\left(X^j\xi^k - \xi^j\bar{\xi}^k\omega\right) + \bar{\xi}^i_l\xi^l\xi^j\bar{\xi}^k\Big], \tag{71}$$

$$
\begin{aligned}
\bar{\epsilon}' = \frac{\epsilon_{ijk}}{4d}\Big\{ & X_r^i\Big[X^j\left(\bar{\xi}^k\omega - 3\xi^k\bar{\omega}\right) - 2\xi^j\bar{\xi}^k U\Big] - X_l^i X^j\left(\xi^k\bar{\xi}^l + \bar{\xi}^k\xi^l\right) \\
& + \xi_r^i\Big[X^j\left[\bar{\xi}^k\left(U + \omega\bar{\omega}\right) - \xi^k\bar{\omega}^2\right] - \xi^j\bar{\xi}^k U\bar{\omega}\Big] \\
& + \xi_l^i\Big[X^j\left(X^l\bar{\xi}^k - \bar{\xi}^l\xi^k\bar{\omega} + \bar{\xi}^l\bar{\xi}^k\omega\right) - \xi^j\bar{\xi}^k\bar{\xi}^l U\Big] \\
& + \bar{\xi}_r^i\Big[X^j\left[\xi^k\left(U + \omega\bar{\omega}\right) - \bar{\xi}^k\omega^2\right] + \xi^j\bar{\xi}^k U\omega\Big] \\
& + \bar{\xi}_l^i\Big[X^j\left(X^l\xi^k - \xi^l\bar{\xi}^k\omega + \xi^l\xi^k\bar{\omega}\right) + \xi^j\bar{\xi}^k\xi^l U\Big] \\
& + \frac{1}{4}\left(2U_r + \bar{\xi}^l\omega_l - \xi^l\bar{\omega}_l + \bar{\omega}\omega_r - \omega\bar{\omega}_r\right), \qquad (72)
\end{aligned}
$$

These relations, for a general null tetrad, generalize the equations of ref. [9] that were derived for a *standard tetrad*.

C Proof of the main theorems

C.1 Proof of Theorem 3

In this first sub-section of the appendix we study the implications of the definition of spacetimes which are *monopolar with angular momentum of type b*. Since the definition is based on the *rotating tetrad*, one has to begin with $\psi_1^0 = 0$. Additionally one requires the Weyl tensor to satisfy $\psi_0 = 0,\ \ \psi_2^0 \neq 0$. In the following proofs we refer to Einstein equations and Bianchi identities as written in reference [9].

The first two radial equations for the spin coefficients are given by Sachs optical scalar equations

$$
\frac{\partial\rho}{\partial r} = \rho^2 + \sigma\bar{\sigma} \quad , \quad \frac{\partial\sigma}{\partial r} = \sigma\left(\rho + \bar{\rho}\right). \qquad (73)
$$

These equations can be written in terms of the matrix $W = \begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \bar{\rho} \end{bmatrix}$ as

$$
\frac{\partial W}{\partial r} = W^2; \qquad (74)
$$

whose solution is given by

$$
W = \begin{bmatrix} \frac{\rho_0 - r}{R^2} & \frac{\sigma_0}{R^2} \\ \frac{\bar{\sigma}_0}{R^2} & \frac{\bar{\rho}_0 - r}{R^2} \end{bmatrix}, \qquad (75)
$$

where $R^2 = (\rho_0 - r)(\bar{\rho}_0 - r) - \sigma_0\bar{\sigma}_0$. In this case ρ_0 is imaginary by the conditions on the coordinate system (see equation (7)).

From the Bianchi identities equations ((3.91) of ref. [9]) one obtains

$$\frac{\partial\psi_1}{\partial r} = 4\rho\psi_1, \tag{76}$$

which implies that $\psi_1 = \psi_1^0 \left(\frac{1}{R^4} + O\left(\frac{1}{R^5}\right)\right)$. However, in the *rotating tetrad* $\psi_1^0 = 0$, and thus $\psi_1 = 0$. Therefore, we have that $\psi_0 = \psi_1 = 0$. Now the Goldberg-Sachs theorem implies that $\sigma = 0$, and therefore ρ becomes

$$\rho = -\left(r + i\,A\right)^{-1}, \tag{77}$$

where $A = -i\,\rho_0$ is a real function of the angular coordinates to be determined.

From equations (3.52), (3.56) and (3.57) of ref. [9] we obtain:

$$\tau = \frac{\tau_0}{r^2 + A^2}, \tag{78}$$

$$\tau' = \frac{-\bar{\tau}_0}{(r + iA)^2}, \tag{79}$$

$$\beta = \frac{\beta_0}{(r - iA)}, \tag{80}$$

$$\beta' = \frac{\bar{\beta}_0}{(r + iA)} - \frac{\bar{\tau}_0}{(r + iA)^2}, \tag{81}$$

where τ_0 and β_0 are, so far, undetermined functions of the angular coordinates. From the torsion free conditions (54) and (56) in the appendix B the radial dependence of the tetrad components ξ^i and X^i is given by:

$$\xi^i = \frac{\xi_0^i}{(r - iA)} \quad , \quad X^i = X_0^i - \frac{\left(\tau_0\bar{\xi}_0^i + \bar{\tau}_0\xi_0^i\right)}{(r^2 + A^2)}, \tag{82}$$

where ξ_0^i are to be determined.

The choice of asymptotic angular coordinates (see equation (10)) requires $X_0^2 = X_0^3 = 0$; while the quantities ξ_0^2 and ξ_0^3 are given by (8) and (9). The value of ξ_0^0 is going to be determined below.

Given a quantity η of type $\{p,q\}$ (see GHP formalism[8]), one can express the action of the ð-operator in terms of the leading order operator $ð_0$ (the ð-operator on the unit sphere), as follows:

$$ð\eta = \frac{ð_0\left(\eta\right)}{\left(r - iA\right)} + q\bar{\tau}^{'}\eta \quad , \quad \bar{ð}\eta = \frac{\bar{ð}_0\left(\eta\right)}{\left(r + iA\right)} + p\tau^{'}\eta. \tag{83}$$

From equation (3.55) of ref. [9] we can determine the angular dependence of the function τ_0 in terms of the function A, as

$$\tau_0 = -ið_0 A. \tag{84}$$

From equation (3.54) of ref. [9] we calculate the value of the spin coefficient σ', namely:

$$\sigma' = \frac{\bar{ð}_0\bar{\tau}_0}{\left(r + i\,A\right)^2}, \tag{85}$$

and from the Bianchi identities (3.89) and (3.90) in ref. [9] we obtain

$$\psi_2 = \frac{\psi_2^0}{\left(r + i\,A\right)^3}, \tag{86}$$

together with

$$ð_0\psi_2^0 = 0. \tag{87}$$

Therefore, we can express ψ_2^0 as $\psi_2^0 = M + iB$ where M and B are two real constants.

From (3.53) of ref. [9] one obtains

$$\rho' = \frac{\rho_0'}{\left(r - iA\right)} - \frac{ð_0\tau_0'}{\left(r^2 + A^2\right)} + \frac{\tau_0\bar{\tau}_0 + r\left(-M - iB\right)}{\left(r + iA\right)\left(r^2 + A^2\right)}. \tag{88}$$

For the case of a *rotating tetrad*, equation (59) of appendix B is an algebraic relation between ρ, ρ' and U; from which one can obtain

$$U = \rho_0' - \frac{\left(ð_0\tau_0' - \bar{ð}_0\bar{\tau}_0'\right)}{2iA} - \frac{\tau_0\bar{\tau}_0}{\left(r^2 + A^2\right)} + \frac{\left(M - \frac{r}{A}B\right)r}{\left(r^2 + A^2\right)}. \tag{89}$$

From equation (3.59) of ref. [9] we obtain $\rho_0' = 1/2$ and

$$2i\left(A + B\right) = ð_0\tau_0' - \bar{ð}_0\bar{\tau}_0'. \tag{90}$$

This last equation implies, using (84),

$$\eth_0\bar{\eth}_0 A = -\left(A + B\right), \tag{91}$$

i.e. A is a combination of spherical harmonics up to $\ell = 1$. Performing a suitable change of coordinates at $\mathcal{I}^+$ one can write A as:

$$A = a\cos\left(\theta\right) - B; \tag{92}$$

moreover, (91) implies $\eth_0\tau_0 = 0$ which together with (85) implies $\sigma' = 0$.

From eq. (3.58) of ref. [9] one obtains

$$\epsilon^{'} = \epsilon_0^{'} + \frac{\left(\bar{\tau}_0\beta_0 - \tau_0\bar{\beta}_0\right)}{\left(r^2 + A^2\right)} + \frac{\tau_0\bar{\tau}_0\left(r - iA\right)}{\left(r^2 + A^2\right)^2} - \frac{M + iB}{2\left(r + iA\right)^2}, \tag{93}$$

and the asymptotic analysis of equation (53) and (58) implies $\epsilon_0' = 0$.

From the torsion free conditions (58), in the leading order of the asymptotic expansion, one obtains

$$\xi_0^0 = -\tau_0 = i\eth_0 A, \tag{94}$$

and from (60), also in the leading order, it is deduced that

$$\beta_0 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial\zeta}P_0, \tag{95}$$

and

$$2iAX_0^0 = \eth_0\bar{\xi}_0^0 - \bar{\eth}_0\xi_0^0. \tag{96}$$

According to equation (10), $X_0^0 = 1$; therefore, the last equation plus (92), and (94) imply $B = 0$.

Straight forward calculation of κ' gives zero. Therefore the null vector n is geodesic with zero shear ($\sigma' = 0$). Then the Goldberg-Sachs theorem implies that $\psi_3 = \psi_4 = 0$. We conclude then that the spacetime is algebraically special of type D.

Finally the value of the tetrad components becomes:

$$\xi^i = \frac{\xi_0^i}{\left(r - ia\cos\left(\theta\right)\right)}, \tag{97}$$

$$X^i = \delta_0^i - \frac{\left(\tau_0\bar{\xi}_0^i + \bar{\tau}_0\xi_0^i\right)}{\left(r^2 + a^2\cos^2\left(\theta\right)\right)}, \tag{98}$$

$$U = -\frac{1}{2} - \frac{\tau_0 \bar{\tau}_0}{(r^2 + a^2 \cos^2(\theta))} + \frac{Mr}{(r^2 + a^2 \cos^2(\theta))}, \tag{99}$$

where $\xi_0^0 = i\eth_0 A$, and $A = a\cos(\theta)$.

This family of algebraic special solutions of type D corresponds to the Kerr family. Notice that our tetrad is different from those appearing in the text books by Hawking[11] and Wald[17] and in references by Newman & Janis[12] and Demianski & Newman[6].

C.2 Proof of Theorem 5

The asymptotic conditions in definition 4 in section 5 for the electromagnetic field allows us to follow the same path in the integration of the fields equation as showed in appendix B for the vacuum case, all up to equation (84), since the Ricci components appearing in the GHP equations up to this point all vanish due to the conditions on the Maxwell field. With these partial results Maxwell equations[15] reduce to:

$$\frac{\partial \phi_1}{\partial r} - 2\rho\phi_1 = 0, \tag{100}$$

$$\eth\phi_1 - 2\tau\phi_1 = 0, \tag{101}$$

$$\frac{\partial \phi_2}{\partial r} - \eth'\phi_1 + 2\tau'\phi_1 - \rho\phi_2 = 0, \tag{102}$$

$$n^a\partial_a\phi_1 - \eth\phi_2 + \tau\phi_2 - 2\rho'\phi_1 = 0. \tag{103}$$

The first three equations can be written in terms of the spin coefficients that where found before equation (84). Equations (100) and (77) imply

$$\phi_1 = \frac{q}{(r + i\,A)^2}; \tag{104}$$

equation (101) tells us that $\eth_0 q = 0$, and so q is a constant. Equations (102), and (77) imply

$$\phi_2 = \frac{\phi_2^0}{r + i\,A}. \tag{105}$$

Equation (103) cannot be solved at this stage since we still don't know the form of the components of the null vector n^a. However, using the stationarity

condition and the asymptotic form of equation (103)[5] we obtain $\eth_0 \phi_2^0 = 0$ which admits $\phi_2^0 = 0$ as the only regular solution (since ϕ_2^0 has spin weight -1 and therefore its expansion in spherical harmonics ${}_sY_{lm}$ contains $l \geq 1$). Therefore,

$$\phi_1 = \frac{q}{(r + i\,A)^2}, \quad \phi_2 = 0, \quad \phi_0 = 0. \tag{106}$$

With this form for the electromagnetic field the rest of the equations in the GHP formalism can be integrated in a similar manner as it was done in the vacuum case. Additional terms appear, but the structure of the equations remains unchanged. Moreover, the function A satisfies the same equation (92) with $B = 0$, i.e., $A = a\cos(\theta)$. The final result is:

$$\xi^i = \frac{\xi_0^i}{(r - ia\cos(\theta))}, \tag{107}$$

$$X^i = \delta_0^i - \frac{\left(\tau_0 \bar{\xi}_0^i + \bar{\tau}_0 \xi_0^i\right)}{(r^2 + a^2\cos^2(\theta))}, \tag{108}$$

$$U = -\frac{1}{2} - \frac{(\tau_0\bar{\tau}_0 + q\bar{q})}{(r^2 + a^2\cos^2(\theta))} + \frac{Mr}{(r^2 + a^2\cos^2(\theta))}, \tag{109}$$

with $\xi_0^0 = i\eth_0(A)$. This null tetrad corresponds to the three parameter family of Kerr-Newman solutions.

References

[1] O. M. Moreschi. Supercenter of mass system at future null infinity. *Class. Quantum Grav.*, 5:423–435, 1988.

[2] O. M. Moreschi and S. Dain. Rest frame system for asymptotically flat space-times. *J. Math. Phys.*, 39(12):6631–6650, 1998.

[3] S. Dain and O. M. Moreschi. General existence proof for rest frame system in asymptotically flat space-time. to appear in: Class. Quantum Grav., 2000.

[4] O. M. Moreschi. On angular momentum at future null infinity. *Class. Quantum. Grav.*, 3:503–525, 1986.

[5]The rest of the terms contained in this equation turn out to reduce to identities ones we complete the integration of the spin coefficients.

[5] O. M. Moreschi. Recent results on rest frames for radiating isolated systems. to appear in proceedings of “Gravitational Waves: a Challenge to Theoretical Astrophysics”, 2000.

[6] M. Demianski and E. Newman. A combined Kerr-NUT solution of the Einstein field equations. *Bull.Acad. Pol. Sci*, XIV(11):653, 1966.

[7] E.T. Newman, and R. Penrose. *J. Math. Phys.*, 3:566, 1962.

[8] R. Geroch, A. Held, and R. Penrose. A space-time calculus based on pairs of null directions. *J. Math. Phys.*, 14:874–881, 1973.

[9] O. M. Moreschi. General future asymptotically flat spacetimes. *Class. Quantum Grav.*, 4:1063–1084, 1987.

[10] J. Goldberg, A. MacFarlane, E. Newman, and F. Rohrlich. *J. Math. Phys.*, 8:2155, 1966.

[11] S. Hawking and G. Ellis. *The large scale structure of space-time*. Cambridge University Press, 1973.

[12] E. T. Newman and A. Janis. Note on the Kerr spinning-particle metric. *J. Math. Phys.*, 6(6):915–917, 1965.

[13] E. T. Newman and R. Penrose. Note on the Bondi-Metzner-Sachs group. *J. Math. Phys.*, 7(5):863–870, 1966.

[14] E. T. Newman and L. Tamburino. *J. Math. Phys.*, 6(6):902, 1962.

[15] E. T. Newman and K. P. Tod. Asymptotically flat space-time. In A. Held, editor, *General Relativity and Gravitation*, volume 2, pages 1–34. Plenum, 1980.

[16] C. Prior. Angular momentum in general relativity. *Proc. R. Soc. Lond.*, A(354):379, 1977.

[17] R. Wald. *General Relativity*. The Chicago University Press, 1984.